DOCUMENT RESUME

ED 055 836 SE 012 431

AUTHOR	Zimmerman, W. Bruce
TITLE	Computer Solution of the Two-Dimensional Tether Ball: Problem to Illustrate Newton's Second Law.
PUB DATE	71
NOTE	16p.; Paper presented at the American Association of Physics Teachers Meeting, February 1971, New York
EDRS PRICE	MF-$0.65 HC-$3.29
DESCRIPTORS	*College Science; *Force; Instruction; Kinetics; Mathematical Applications; *Motion; *Physics; Secondary School Science; *Teaching Techniques

ABSTRACT

Force diagrams involving angular velocity, linear velocity, centripetal force, work, and kinetic energy are given with related equations of motion expressed in polar coordinates. The computer is used to solve differential equations, thus reducing the mathematical requirements of the students. An experiment is conducted using an air table to check theoretical considerations with actual results which were in close agreement with predicted values. (TS)

COMPUTER SOLUTION OF THE TWO-DIMENSIONAL TETHER BALL:

PROBLEM TO ILLUSTRATE NEWTON'S SECOND LAW

W. Bruce Zimmerman*

Andrews University

Berrien Springs, Michigan 49104

Application of the laws of physics to phenomena that the student is familiar with is often interesting, informative, and instructive. However, many such problems cannot be solved in closed form, but must be solved numerically. The computer makes such problems relatively easy to solve. Most students are familiar with the game of tether ball and have observed the increase in the angular speed of the ball as the string wraps around the pole. The intuitive feeling is usually that the linear speed of the ball also increases.

A 2-dimensional tether ball which moves on a frictionless table simplifies the problem, but still allows investigation of the physical quantities of interest. The first slide (Slide 1--force diagram) shows the essentials of this 2-dimensional problem. The position of the ball of mass given in polar coordinates r and ϕ with the origin at the center of the pole of radius a, the pole being perpendicular to the table. The only force exerted on the ball is the force $\underline{f}$ arising from the string as it wraps around the pole. The unit vectors $\underline{u}_r$ and $\underline{u}_\phi$ indicate the positive directions of r and ϕ, respectively.

*Present address: Department of Physics, Indiana University at South Bend, South Bend, Indiana, 46615

If the student now resolves the force $\underline{f}$ into its components in the r and ϕ directions, he notes a disquieting thing. Namely, the force component f_ϕ points in the $-\underline{u}_\phi$ direction, opposite to the motion of the ball. The deduction he makes from this fact is that the ball should slow up in the ϕ-direction, rather than increase in angular speed as it moves around the pole. Another interesting fact that emerges from inspection of the diagram is that the force $\underline{f}$ is always perpendicular to the path followed by the ball, and therefore the work done by the force $\underline{f}$ is zero through out the motion of the ball. If the student is able to recall the work-energy theorem, namely, that the work done by the resultant force is equal to the change in kinetic energy, he obtains the result that the change in kinetic energy between any two points on the path is zero. Therefore, the kinetic energy of the ball must be a constant, and it then follows that the linear speed of the ball must also be a constant.

The student is now told to have faith in Newton's 2nd law. The ball is not microscopic in size nor is it going to be traveling at a speed near that of light. Therefore, the answers to all questions relating to the motion of the ball, should be obtained by correctly applying Newton's 2nd law to the problem. To proceed analytically, the reasonable assumption is made that the force $\underline{f}$ is a centripetal-like force, the point where the string is just touching the pole acting as the instantaneous center of the "circular" motion. The expression for the magnitude of this force is shown on the next slide (Slide 2--equations). Even though the speed v has been deduced to be a constant, the point of view is taken that this fact should emerge as a result from applying Newton's 2nd law. Thus, v^2 is replaced by its polar coordinate expression. Resolving the force f into its components, and after some manipulation and simplification, equations of motion in the r and ϕ directions are obtained, as shown on the lower part of the slide. Both equations are nonlinear. Although the equation in r is simple enough so that it can be solved in closed form by defining a new variable (although it is not obvious what that new variable should be: $u = 2-r^2$), the one in ϕ is not. Since the use of the computer to solve differential equations is to be emphasized in this problem, both equations are solved numerically using Euler's method for simplicity.

It is useful to direct the student's attention to these equations before proceeding to the numerical solution. First of all, it is noted that the mass of the ball does not appear, and so is unimportant as far as the motion is concerned. The equation in r simple. It says that $\ddot{r}$ is always negative. Therefore $\dot{r}$ is always negative also, in agreement with the fact that the ball always moves toward the center of the pole. The equation in ϕ is more complicated. At first glance it would appear that $\ddot{\phi}$ is always negative indicating that the angular velocity $\dot{\phi}$ should become smaller, disagreeing with experience, but agreeing with the fact that f_ϕ points in the $-\phi$ direction. Inspecting the equation more closely, however, it is seen that although the 1st two terms on the right hand side are always negative, the 3rd term is not. Because $\ddot{r}$ is negative, this latter term will be positive, assuming that $\dot{\phi}$ has a positive initial value. It turns out that this 3rd term is the predominant one, and is always larger than the sum of the 1st two terms. For this reason $\ddot{\phi}$ is always positive, and the angular velocity $\dot{\phi}$ indeed increases. The 1st two terms arise because of the f_ϕ force, and it is seen that they do try to slow down the motion in ϕ. However, this effect is masked by the 3rd term, which is a kinematical-type term. It has to do with the fact that a given arc length subtends larger and larger angles as the origin of the co-ordinate system is approached.

As a by-product of the numerical solution using the computer, the actual values of these 3 terms can easily be printed out as the solution is built up numerically. It is very illuminating to the student to actually see that the 1st two terms are always negative and small, and that the 3rd term is positive and relatively large.

The initial conditions used in this example are shown in the next slide (Slide 3--initial conditions). Graphs of the results are shown in the next several slides. This slide (Slide 4--r and $\dot{r}$ vs t) shows the behavior of r and $\dot{r}$ or v_r versus t. The next slide (Slide 5--ϕ and $\dot{\phi}$ vs t) shows the variation of ϕ and the angular velocity $\dot{\phi}$ or ω. We see that ω does increase as the ball is drawn nearer the pole as time goes on (in fact, it increases more and more rapidly), in accord with our experience.

The next slide (Slide 6--f vs t) shows what happens to the force magnitude as time goes on. It appears that both $\dot{\phi}$ and f blow up as the ball approaches the pole. In actual practice this behavior does not occur because the ball is not a particle, as we have assumed, but has a non-zero size and so hits the pole while $\dot{\phi}$ and f still have finite values.

The next slide (Slide 7--Work vs t) is a very interesting one. It shows the work W_r and W_ϕ performed in the r and ϕ-directions, respectively, as well as the total work done. We can now see why the f_ϕ force component is necessary--namely, to give a negative work to just balance the positive work done by the f_r force component. The total work done is therefore zero, agreeing with our observation that the force $\underline{f}$ is always perpendicular to the path, and so can do no work. To obtain actual numerical values from the computer giving these results for W_r, W_ϕ, and W is quite striking.

The next slide (Slide 8--path of ball) shows the actual path followed by the ball, and its position at equal intervals of time (time interval = 10 msec). The even spacing of the dots reflects the constant linear speed that the ball has, which is consistent with the result obtained earlier from the work-energy theorem. Again the computer printout of the same values for the speed as the initial speed (in this case, 100 cm/sec) is also quite striking.

As good physicists should, it is desirable to check the theoretical results with actual experiment. It is not difficult to set this experiment up on an air table and photograph the position of the mass using a strobe. The next slide (slide 9--strobe photo)* shows the result. The similarity between this slide and the previous one is quite evident. Careful measurement shows that the puck does slow down slightly, but this behavior is apparently due to the small frictional force that exists between the air layer and puck.

The problem can also be set up to show that as the size of Δt in the numerical analysis is made smaller, the accuracy of the answers improve. But only up to a point, for computers have a limit to their word length and round off error also plays a role. Thus, some of the pitfalls of numerical analysis can also be displayed.

My experience has shown that problems like this one are instrumental in building up the student's confidence in understanding and applying the laws of physics, especially where it appears that the formal application of the physical laws is going to yield incorrect answers, and then doesn't.

In closing, I would like to thank Mr. Bruce Lee for sugesting the problem, and for many helpful discussions.

*Slide 9 could not be reproduced.

APPENDIX--Derivation of the Equations of Motion

The diagram in Figure 1 shows the position of the mass m at a time t, and the pertinent variables in an (r,ϕ) polar coordinate system with origin at the center of the pole. The radius of the pole is a, the initial length of the string is ℓ_0, $\underline{v}$ is the [illegible] city of the ball, $\underline{f}$ is the force exerted on the ball by the string, and ϕ' is the angle between the horizontal and the radial line drawn to the point C where the string is just beginning to make contact with the pole.

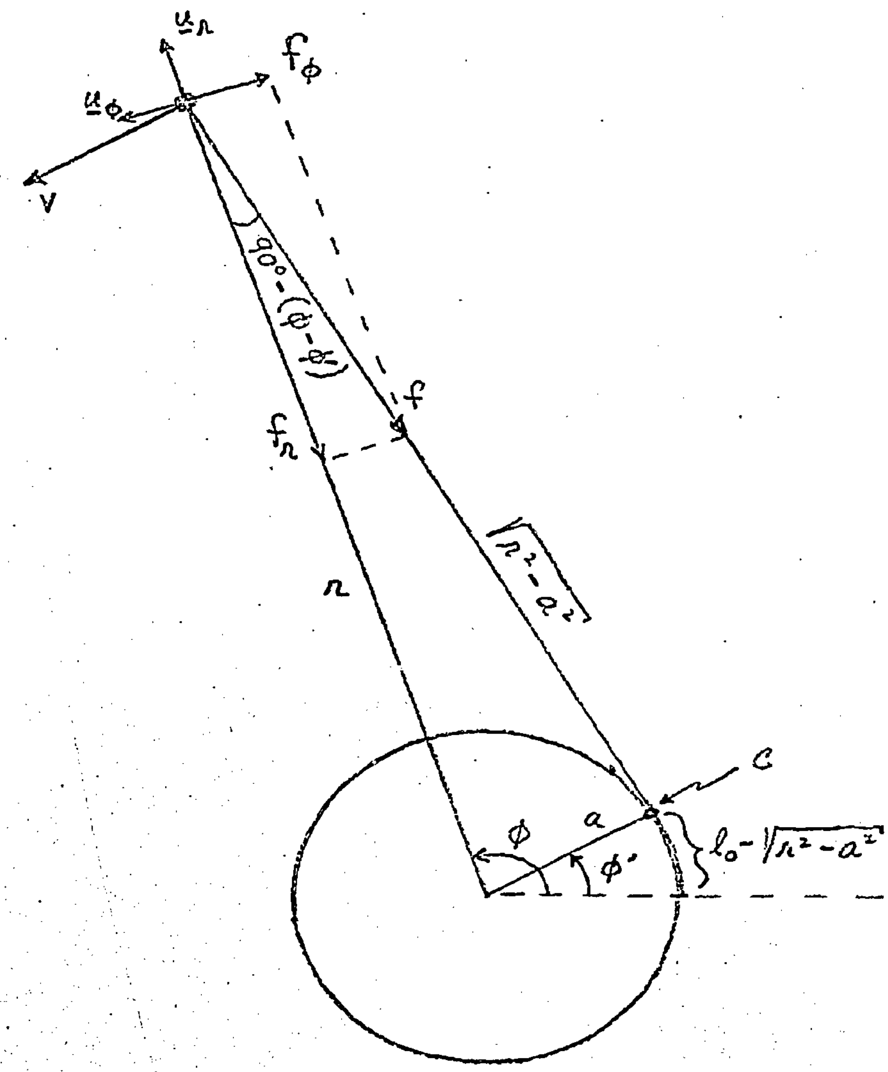


Figure 1

Assuming that the force $\underline{f}$ is a centripetal one with instantaneous center of rotation at C, we have for its magnitude

$$(1) \qquad f = \frac{mv^2}{\sqrt{r-a^2}} = \frac{m(\dot{r}^2 + r^2\dot{\phi}^2)}{\sqrt{r^2-a^2}}.$$

The components are seen to be

$$(2)\begin{cases} f_r = -f\cos\left[90^\circ-(\phi-\phi')\right] = -\dfrac{m(\dot r^2+r^2\dot\phi^2)}{\sqrt{r^2-a^2}}\sin(\phi-\phi') \\ f_\phi = -f\sin\left[90^\circ-(\phi-\phi')\right] = -\dfrac{m(\dot r^2+r^2\dot\phi^2)}{\sqrt{r^2-a^2}}\cos(\phi-\phi') \end{cases}$$

Inspection of Figure 1 shows that

$$(3)\qquad \sin(\phi-\phi') = \frac{\sqrt{r^2-a^2}}{r}, \qquad \cos(\phi-\phi') = \frac{a}{r}.$$

Substituting equations (4) into equations (2) gives

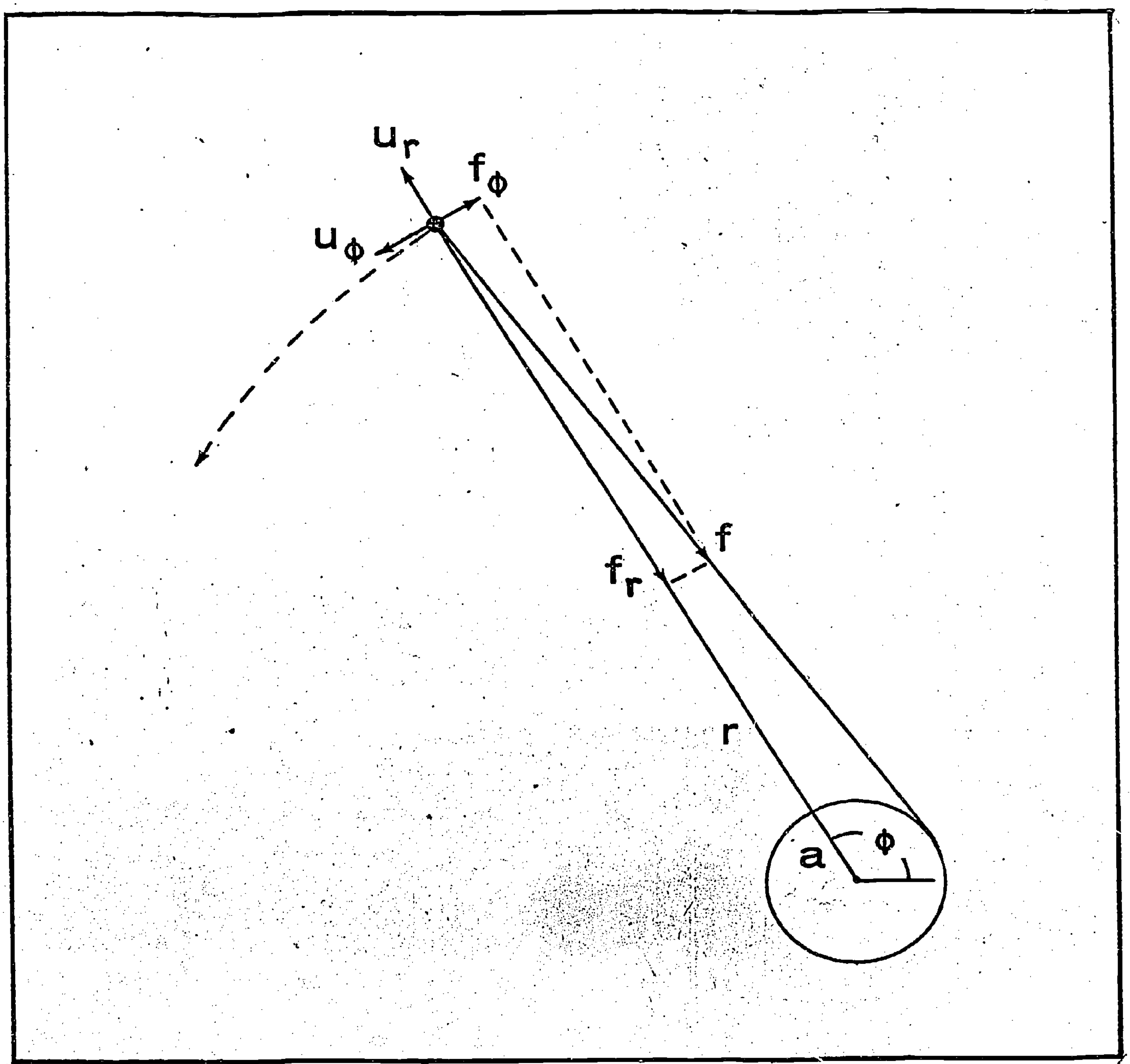
$$(4)\begin{cases} f_r = -\dfrac{m(\dot r^2+r^2\dot\phi^2)}{r} \\ f_\phi = -\dfrac{ma(\dot r^2+r^2\dot\phi^2)}{r\sqrt{r^2-a^2}} \end{cases}$$

Writing Newton's 2nd law in polar coordinates gives

$$(5)\begin{cases} m(\ddot r - r\dot\phi^2) = f_r \\ m(r\ddot\phi + 2\dot r\dot\phi) = f_\phi \end{cases}$$

Substituting equations (4) into (5), and simplifying yields the equations of motion:

$$(6)\begin{cases} \ddot r = -\dfrac{\dot r^2}{r} \\ \ddot\phi = -\dfrac{\dot r^2}{r^2\sqrt{\left(\frac{r}{a}\right)^2-1}} - \dfrac{\dot\phi^2}{\sqrt{\left(\frac{r}{a}\right)^2-1}} - \dfrac{2\dot r\dot\phi}{r} \end{cases}$$



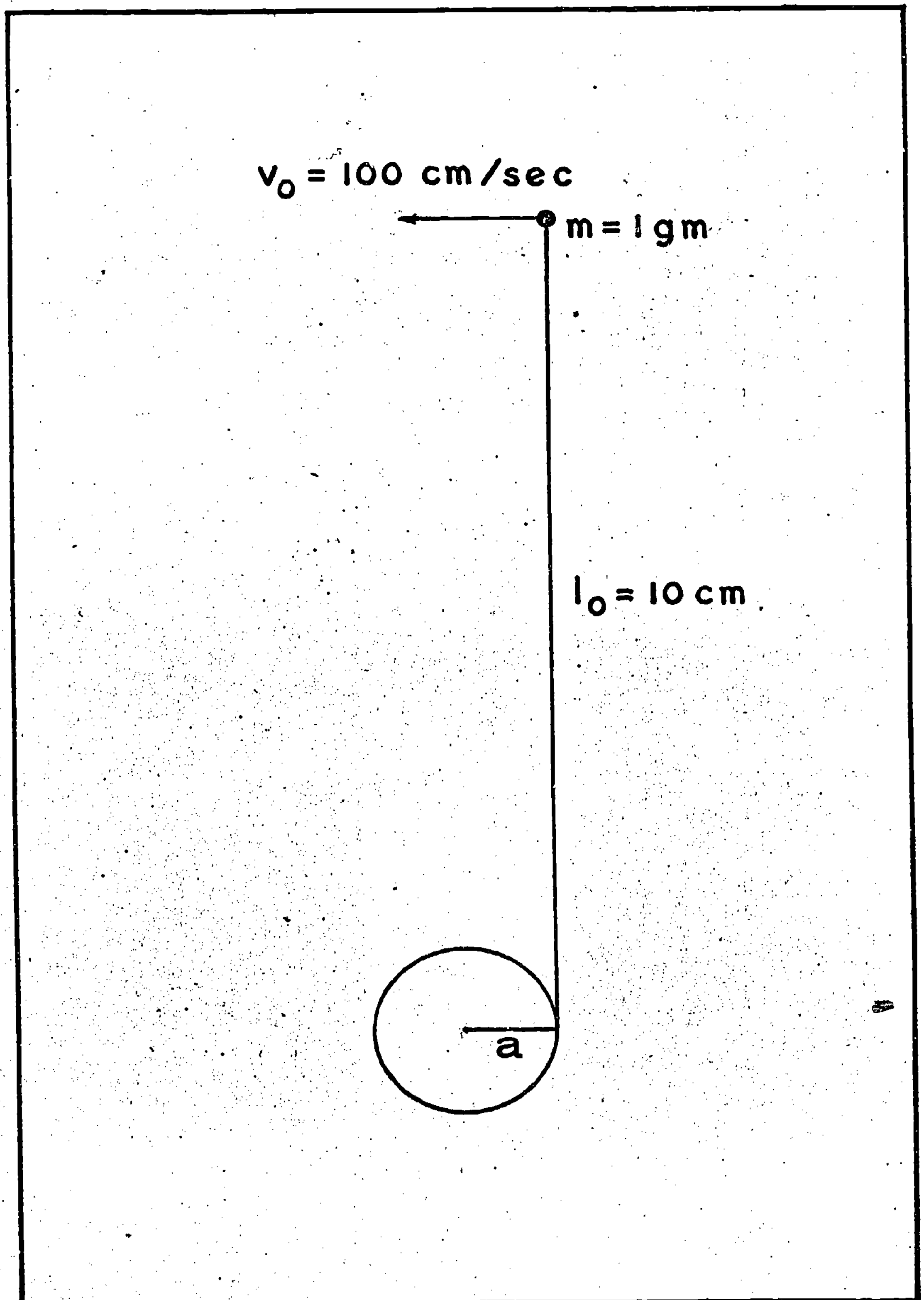
Drawing for Slide 1

$$f = \frac{mv^2}{\sqrt{r^2 - a^2}} = \frac{m(\dot{r}^2 + r^2\dot{\phi}^2)}{\sqrt{r^2 - a^2}}$$

$$\ddot{r} = -\frac{\dot{r}^2}{r}$$

$$\ddot{\phi} = -\frac{\dot{r}^2}{r^2\sqrt{\left(\frac{r}{a}\right)^2 - 1}} - \frac{\dot{\phi}^2}{\sqrt{\left(\frac{r}{a}\right)^2 - 1}} - \frac{2\dot{r}\dot{\phi}}{r}$$

Copy for Slide 2



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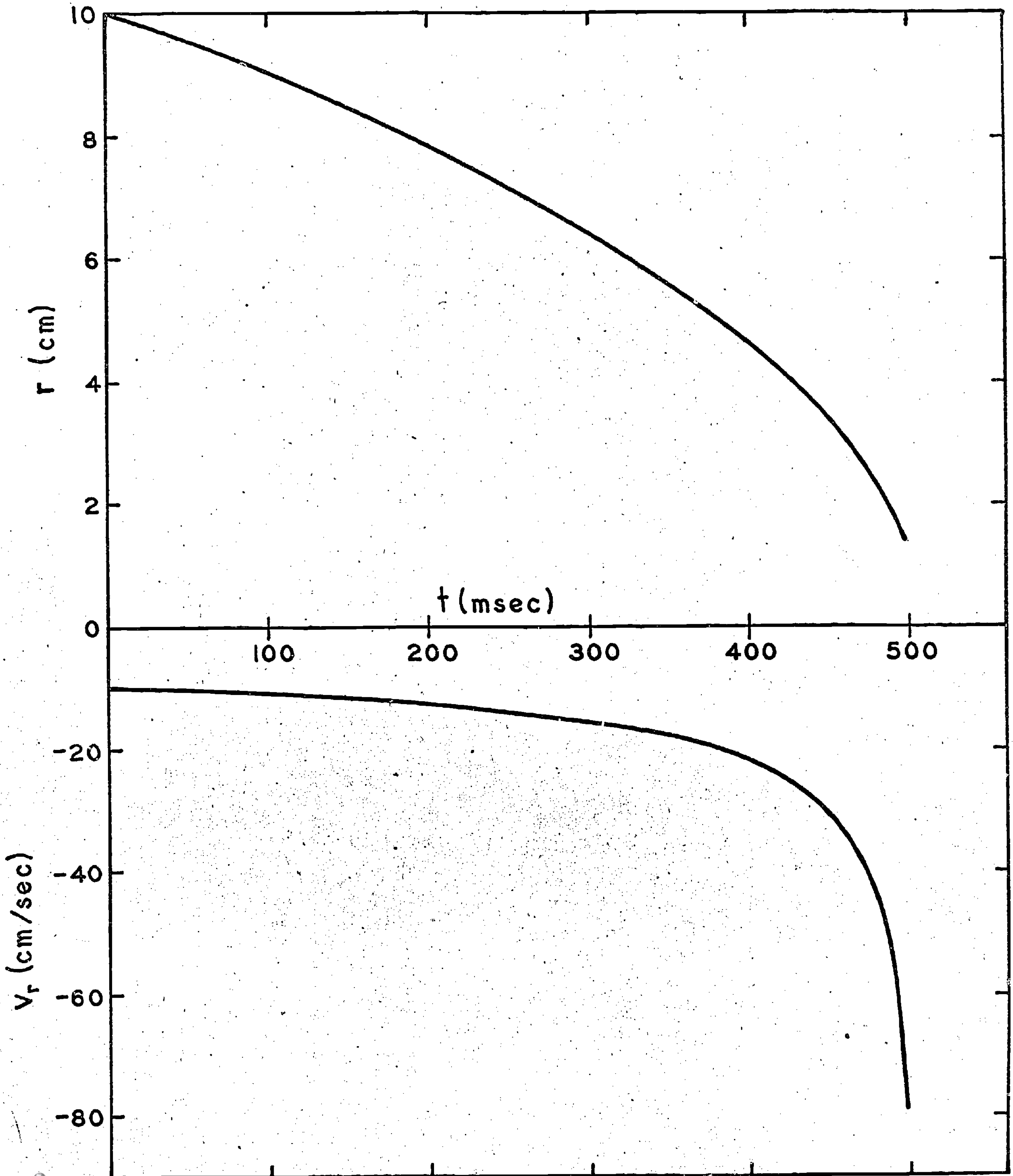

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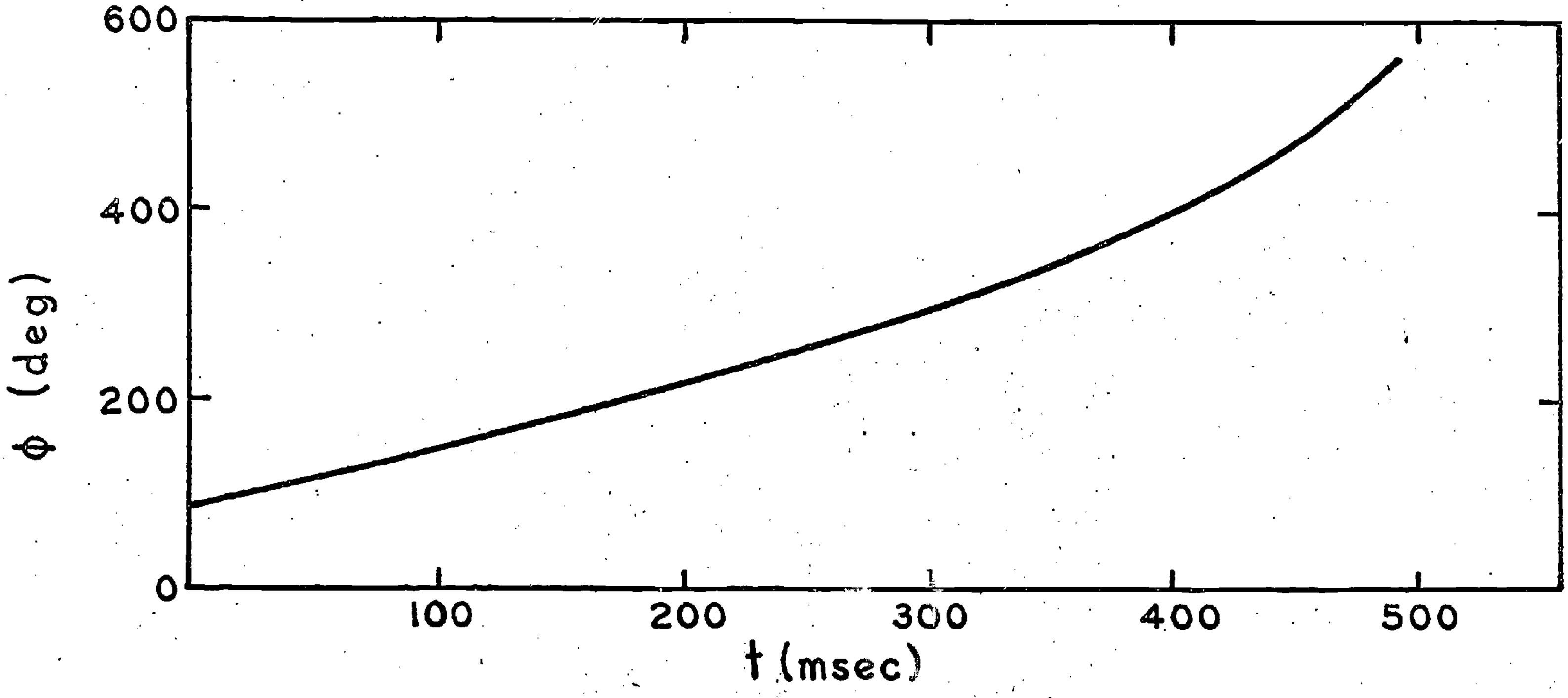

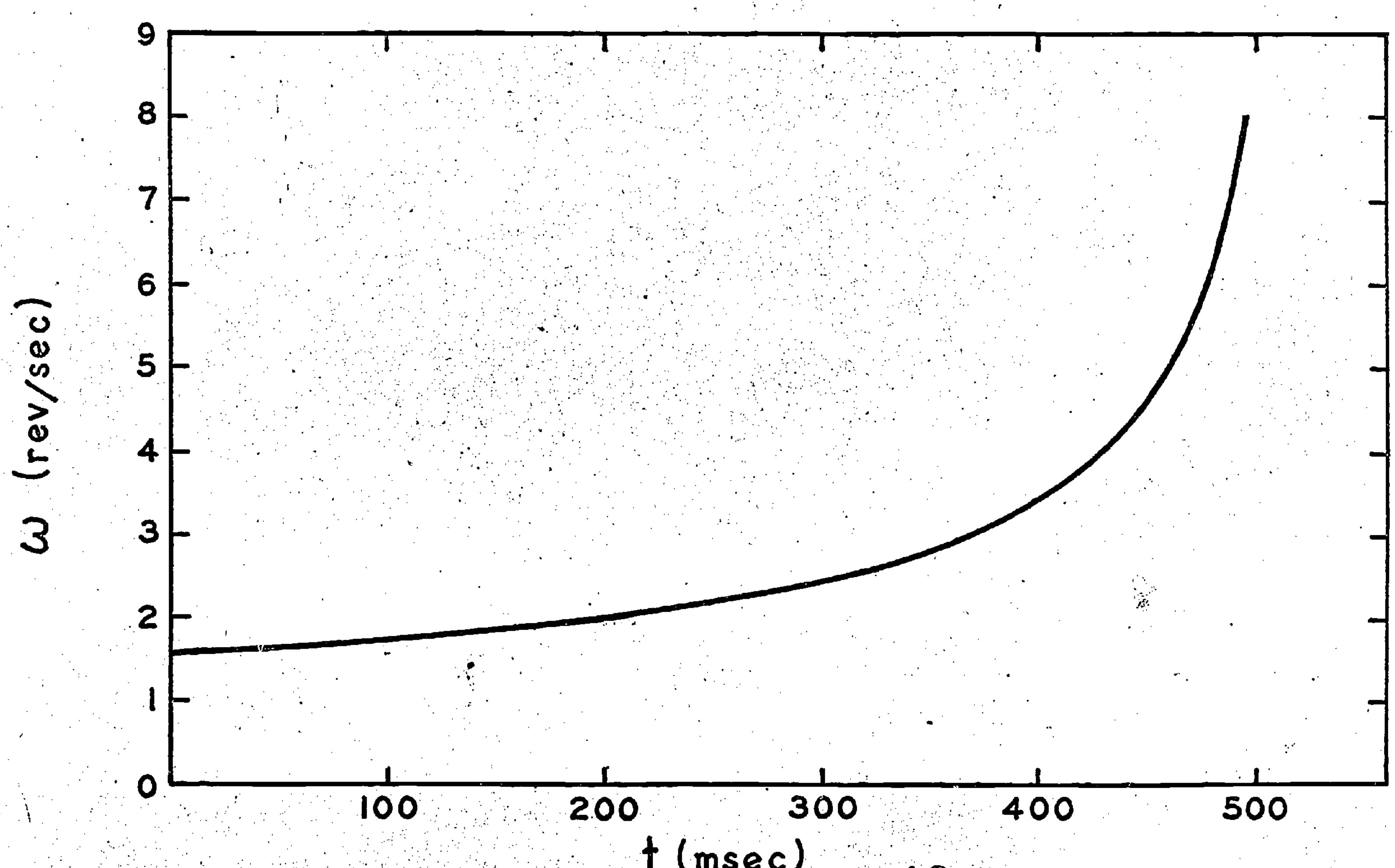

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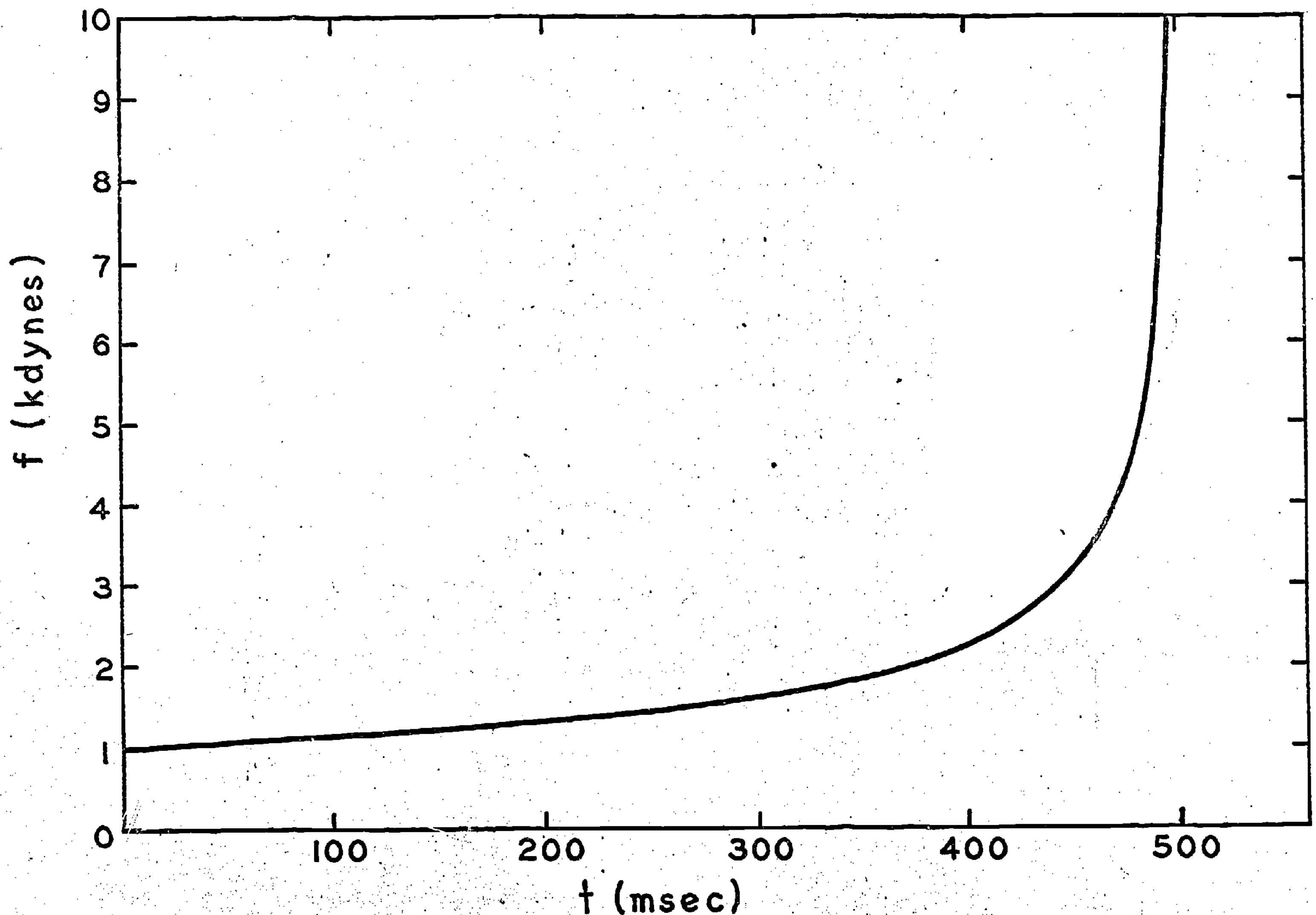

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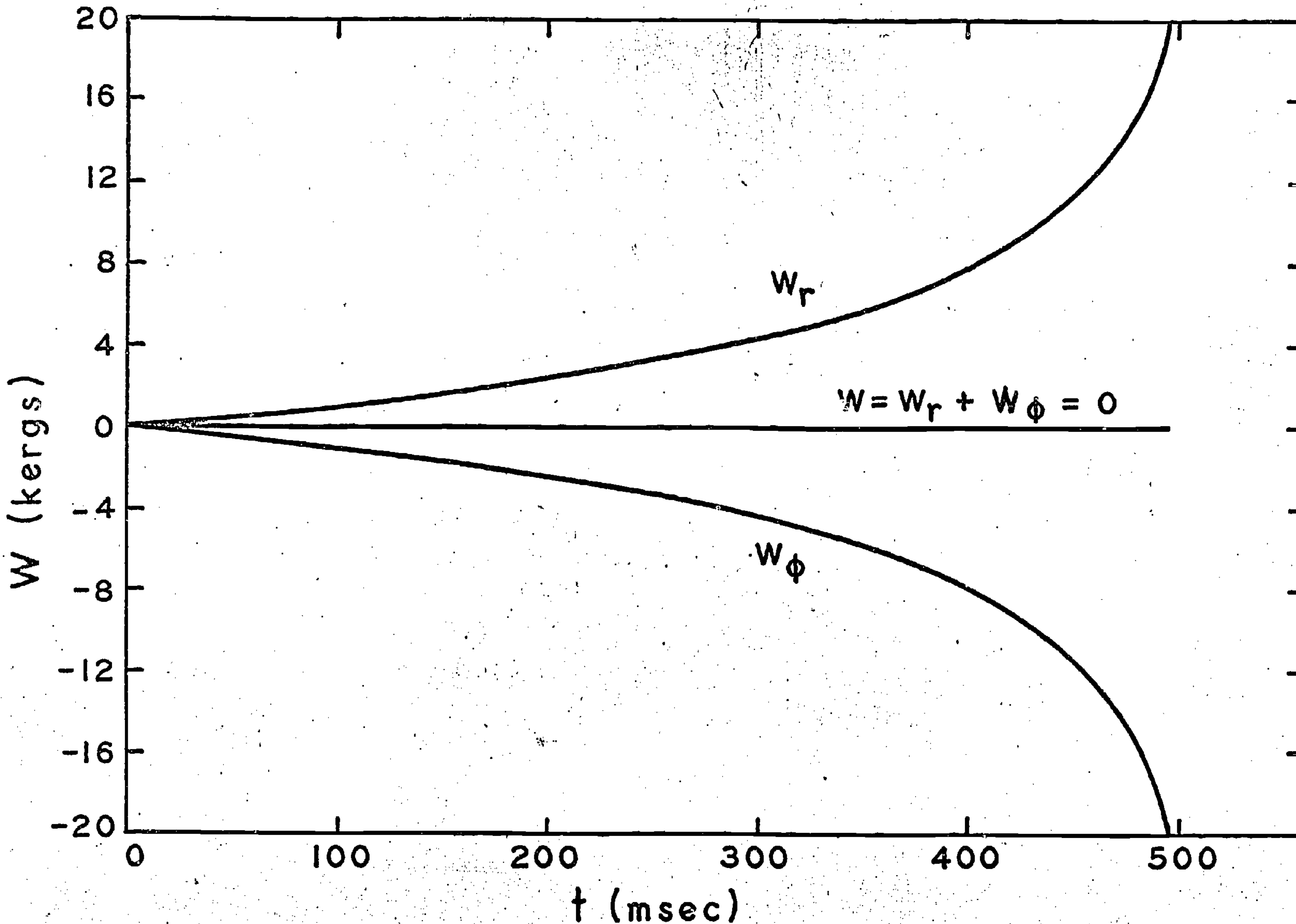

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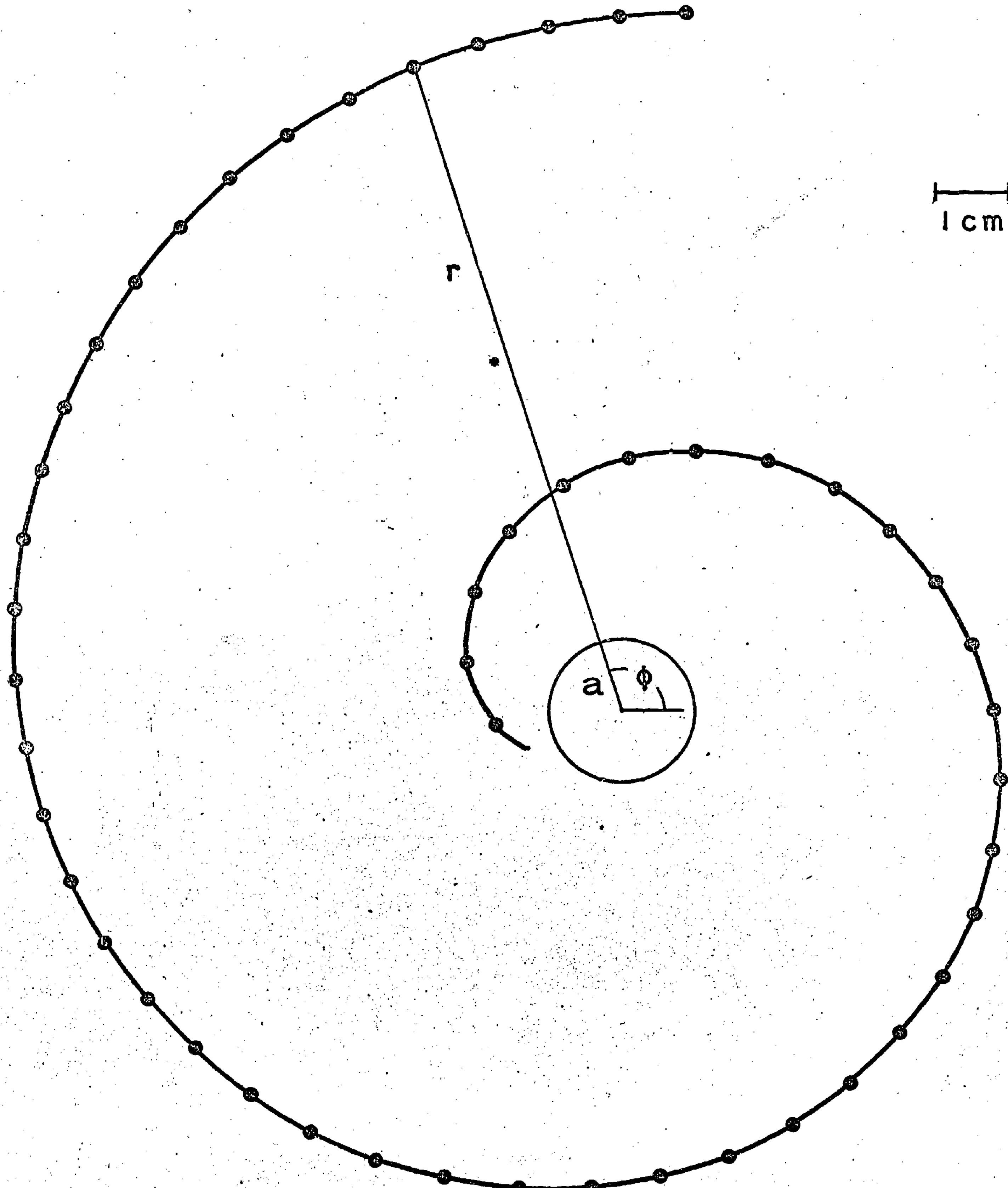

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